PHYSICS OF THE FORMATION OF FIBER LIGHTGUIDES

M. E. Zhabotinskii and A. V. Foigel'

UDC 532.521

A fiber lightguide is a fine continuous or tubular transparent filament. Fiber lightguides are formed from the liquid mass exuded through a dye or drawn from a suitable blank. Both of these processes can be considered using the equations of the hydrodynamics of an incompressible Newtonian liquid. (Polymers, which are not Newtonian liquids, are not considered here.) The drawing of a continuous glass fiber from a dye is considered in [1]. The drawing of a microcapillary from a dye is considered in [2], where a qualitative consideration is given which is insufficient for an understanding of the effect of different parameters of the process on the dimensions of the drawn microcapillary. In this paper we consider the formation of a microcapillary from a tubular blank using the approximation of an incompressible Newtonian liquid with a variable viscosity determined by the given temperature distribution. The effect of surface tension and of the excess pressure produced in the channel to counteract the surface tension are taken into account. It is assumed that the drawing process is steady, the blank has thin walls and is axisymmetrical, and the transition to a microcapillary occurs smoothly. With these assumptions the problem of obtaining the shape of the transition and the dimensions of the microcapillary obtained is reduced to a system of ordinary differential equations. The dependence of the dimensions of the microcapillary on the dimensions of the blank and the parameters of the process is established, thereby enabling the process to be optimized.

The process by which a microcapillary is formed is described by the Navier–Stokes equations and by the equation of continuity. The viscosity of the blank and the finished microcapillary is assumed to be infinitely large, it is a known function of the temperature, the temperature distribution is given, the liquid is isotropic, and its motion is assumed to be axisymmetric, which makes the problem two-dimensional.

We are given the thickness of the wall h_0 and the mean radius $\bar{r}_0$ of the blank, its feed rate u_0, and the microcapillary drawing speed u_∞ (Fig. 1). In the solution we take into account the surface tension σ and the pressure drop $\Delta p = p_1 - p_2$ between the channel and the external medium, both these quantities being assumed constant along the z axis.

The blank and the finished microcapillary are assumed to be relatively thin-walled, i.e., over the whole cross section the thickness of the wall is assumed to be small compared with the radius. Assuming that heat transfer to the blank inside the heater and subsequent cooling of the fiber are due to radiation, and the absorption coefficient of the material is small, the temperature of the liquid will be constant at all points of the transverse cross section of the jet. This means that the temperature distribution and, consequently, the viscosity depend only on the longitudinal coordinate and are described by the given function $\eta(z)$. It is clear from the formulation of the problem that $\eta(z)$ is a smooth function which approaches infinity as $z \to \pm\infty$ (see Fig. 1).

We wish to find the shape of the jet (the transition from the blank to the microcapillary), i.e., the thickness of the wall and the radius of the jet as a function of the longitudinal coordinate, including the thickness of the wall h_∞ and the mean radius $\bar{r}_\infty$ of the drawn microcapillary.

As was done in [1], we change from the Navier–Stokes equations

Moscow. Translated from Zhurnal Prikladnoi Mekhaniki i Tekhnicheskoi Fiziki, No. 2, pp. 167-174, March-April, 1976. Original article submitted March 24, 1975.

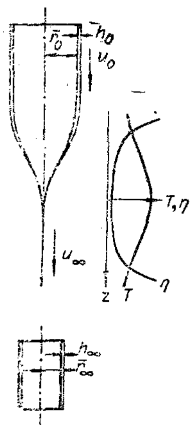

Fig. 1

$$\partial\Pi_{ik}/\partial x_k = 0,\ i,\ k = 1,\ 2,\ 3,$$
$$\Pi_{ik} = \rho v_i v_k - \sigma_{ik},\ \sigma_{ik} = -p\delta_{ik} + \eta(\partial v_i/\partial x_k + \partial v_k/\partial x_i), \tag{1}$$

where Π_{ik} is the tensor of the momentum flux density, σ_{ik} is the stress tensor, v_i and x_i are the components of the velocity and the coordinate, p is the pressure, ρ is the density, and η is the viscosity, to the momentum equation which is obtained by integrating Eq. (1) over the volume of the liquid contained between the sections $z = z_1$ and $z = z_2$. We will change from a volume integral to a surface integral:

$$\oint \Pi_{ik} df_k = 0, \tag{2}$$

where df_k is an element of the surface and the integration is carried out over the closed surface consisting of the two transverse cross sections at $z = z_1$ and $z = z_2$ and the two side surfaces: the internal surface $[r = r_1(z)]$ and the external surface $[r = r_2(z)]$. The boundary conditions on the side surfaces have the form

$$\sigma_{zz}\sin\theta_{1,\,2} + \sigma_{rz}\cos\theta_{1,\,2} = -\,[p_{1,\,2} \mp (\sigma\cos\theta_{1,\,2}/r_{1.2})]\sin\theta_{1,\,2}; \tag{3}$$
$$\sigma_{rr}\cos\theta_{1,\,2} + \sigma_{rz}\sin\theta_{1,\,2} = -\,[p_{1,\,2} \mp (\sigma\cos\theta_{1,\,2}/r_{1,2})]\cos\theta_{1,\,2}, \tag{4}$$

where $\theta_{1,2}$ are the angles of inclination of the internal and external boundaries in the $r-z$ plane

$$\operatorname{tg}\theta_{1,2} = -dr_{1,2}(z)/dz;\ \ dr_{1.2}(z)/dz = v_r/v_z|_{r=r_{1.2}(z)}. \tag{5}$$

Substituting the tensor Π_{ik} into the z-th component of Eq. (2), neglecting the nonlinear terms proportional to the density in view of the smallness of the Reynolds number, substituting Eq. (3) into the integral over the internal and external side surfaces, and assuming the angles of inclination $\theta_{1,2}$ to be small so that $\theta_{1,2} \approx 1$, we obtain

$$\int_{r_1(z_2)}^{r_2(z_2)} \sigma_{zz} r dr - \int_{r_1(z_1)}^{r_2(z_1)} \sigma_{zz} r dr = -\int_{z_1}^{z_2}\left[r_2(z)\left(p_2 + \frac{\sigma}{r_2(z)}\right)\frac{\partial r_2(z)}{dz} - r_1(z)\left(p_1 - \frac{\sigma}{r_1(z)}\right)\frac{\partial r_1(z)}{dz}\right]dz. \tag{6}$$

Since the surface tension σ and the pressure of the air p_1 and p_2 are assumed constant, the integral on the right side of Eq. (6) can be taken in parts:

$$\int_{r_1(z)}^{r_2(z)}\left[p - 2\eta(z)\frac{\partial v_z}{\partial z}\right] r dr = \frac{1}{2}\left[p_2 r_2^2(z) - p_1 r_1^2(z)\right] + \sigma\,[r_1(z) + r_2(z)] + c. \tag{7}$$

We will assume that for all z

$$(r_2 - r_1)\partial p/\partial r \ll p,\ \ (r_2 - r_1)\,\partial^2 v_z/\partial r\partial z \ll \partial v_z/\partial z \tag{8}$$

[after obtaining the solution we will find by direct substitution into the Navier–Stokes equations under what conditions relations (8) are satisfied]. The pressure p and the longitudinal component of the velocity v_z can then be assumed constant over each cross section $z = \text{const}$ and to be dependent only on the longitudinal coordinate z. Taking the expression in square brackets in Eq. (7) outside the integral we obtain

$$\left[r_2^2(z) - r_1^2(z)\right][p - 2\eta(z)(dv_z/dz)] = \left[p_2 r_2^2(z) - p_1 r_1^2(z)\right] + 2\sigma\,[r_1(z) + r_2(z)] + 2c.$$

Integrating with respect to the variable r the equation of continuity

$$\operatorname{div} v = 0$$

and taking into account the above assumptions, we obtain

$$v_r = -(r/2)dv_z/dz + A(z)/r, \tag{9}$$

where A(z) is an unknown function. We use the boundary conditions (4) to find p(z) and A(z), we substitute the expression for the radial velocity (9) into the equations of the boundaries (5), and we integrate them. Then, changing from the inner and outer radii $r_1(z)$ and $r_2(z)$ to the wall thickness h(z) and the mean radius $\bar{r}(z)$

$$h(z) = r_2(z) - r_1(z);\ \bar{r}(z) = (1/2)[r_1(z) + r_2(z)]$$

and putting $u = v_z$, $\Delta p = p_1 - p_2$, we obtain a set of equations with the boundary conditions

$$\eta du/dz - \alpha u = -\sigma/3h; \tag{10}$$

$$2udh/dz + hdu/dz = \sigma/\eta - (\bar{r}\Delta p/2\eta)[1 - (h^2/4\bar{r}^2)]; \tag{11}$$

$$\bar{r}hu = \beta, \tag{12}$$

$$u|_{z=-\infty} = u_0;\ \bar{r}|_{z=-\infty} = \bar{r}_0;\ h|_{z=-\infty} = h_0;\ u|_{z=+\infty} = u_\infty, \tag{13}$$

where α and β are unknown constants, u_0 and u_∞ are the feed rate and the drawing velocity, and h_0 and $\bar{r}_0$ are the thickness of the wall and the mean radius of the blank.

From a comparison of Eq. (12) with $z = -\infty$ and Eqs. (13) we obtain that $\beta = \bar{r}_0 h_0 u_0$. The assumption that the walls of the fiber are thin enables us to neglect the quantity $(h/\bar{r})^2$ in Eq. (11) compared with unity, and after substituting the value of h from Eq. (12) into Eqs. (10) and (11) we obtain two differential equations for $\bar{r}$ and u and an explicit expression for h.

Changing to the dimensionless variables

$$Z = z/l,\ H = h/h_0,\ R = \bar{r}/\bar{r}_0,\ U = u/u_0,\ \mu = \eta_0/\eta, \tag{14}$$

where η_0 is the minimum viscosity, and the effective length of the heating zone l is given by the equation

$$l/\eta_0 = \int_{-\infty}^{\infty} dz/\eta(z); \tag{15}$$

the equations and the boundary conditions will now contain the dimensionless parameters

$$\begin{gathered} U_\infty = u_\infty/u_0,\ w = \ln U_\infty,\ \gamma = \alpha l/\eta_0 w, \\ P = \bar{r}_0 \Delta p l/2\eta_0 u_0 h_0 w,\ Q = \sigma l/\eta_0 u_0 h_0 w. \end{gathered} \tag{16}$$

In the new variables the equations and boundary conditions will have the form

$$dU/dz - \gamma\mu w U = -(Q/3)\mu w R U; \tag{17}$$

$$2UdR/dz + RdU/dz = -\mu w R^2 U(Q - PR); \tag{18}$$

$$U|_{z=-\infty} = 1;\ R|_{z=-\infty} = 1, \tag{19}$$

$$U|_{z=+\infty} = U_\infty;\ H = 1/RU. \tag{20}$$

The constant γ is found from the condition for the boundary conditions (19) and (20) to be satisfied, the number of which exceeds by one the order of the system (17) and (18).

When $P = Q = 0$, which corresponds to zero surface tension and zero pressure drop ($\Delta p = \sigma = 0$), Eqs. (17) and (18) when $\gamma = 1$ have a solution which satisfies all the boundary conditions

$$U^{(0)}(Z) = \exp(ws(Z));\ R^{(0)}(Z) = [U^{(0)}(Z)]^{-1/2}; \tag{21}$$

$$H^{(0)}(Z) = [U^{(0)}(Z)]^{-1/2}, \tag{22}$$

where

$$s(Z) = \int_{-\infty}^{Z} \mu(\zeta)\, d\zeta. \tag{23}$$

To solve Eqs. (17) and (18) when $P^2 + Q^2 \neq 0$ we change from the variable Z to the variable s(Z). In this case, the interval $(-\infty, \infty)$ of variation of Z corresponds to the interval (0.1) of variation of s, since as follows from Eqs. (14) and (15), $s(+\infty) = \int_{-\infty}^{\infty} \mu(\zeta)d\zeta = 1$. Substitution of Eq. (23) into Eqs. (17) and (18) and arithmetic transformation lead to a system of equations with separable variables:

$$dR/ds = -(wR/2)[\gamma + (2/3)QR - PR^2]; \tag{24}$$

$$(1/U)dU/dR = (2/3)(QR - 2\gamma)/[\gamma R + (2/3)(QR^2 - PR^3)] \tag{25}$$

with the boundary conditions

$$U|_{s=0} = 1; \; R|_{s=0} = 1; \tag{26}$$

$$U|_{s=1} = \exp(w). \tag{27}$$

Integrating Eqs. (24) and (25) with the boundary condition (26) we obtain for R and U as implicit functions of the independent variable s expressions containing the unknown constant γ:

$$[y_1/(y_1 - y_2)]\ln[(1 - Ry_1)/R(1 - y_1)] + [y_2/(y_2 - y_1)]\ln[(1 - Ry_2)/R(1 - y_2)] = \gamma ws/2; \tag{28}$$

$$\ln U = [(3y_1 + y_2)/(y_1 - y_2)]\ln[(1 - Ry_1)/(R(1 - y_1)] + [(3y_2 + y_1)/(y_2 - y_1)]\ln[(1 - Ry_2)/R(1 - y_2)], \tag{29}$$

where $y_{1,2}$ are quantities which are inverse to the roots of the quadratic trinomial on the right side of Eq. (24)

$$y_{1,\,2} = (-Q \pm \sqrt{Q^2 + 9P\gamma})/3\gamma. \tag{30}$$

To obtain the unknown constant γ we use the boundary condition (27). Substituting Eq. (27) into Eqs. (28) and (29) and eliminating from the equations obtained the unknown quantity R(1), we obtain that $\gamma = \gamma(P, Q, w)$ is the root of the equation

$$\begin{gathered}(1 - y_2)\exp[w(\gamma - 1)(3y_1 + y_2)/2(y_1 + y_2)] - (1 - y_1) \times \\ \times \exp[w(\gamma - 1)(3y_2 + y_1)/2(y_1 + y_2)] = (y_1 - y_2)\exp(-w/2)\end{gathered} \tag{31}$$

[according to Eq. (30) γ occurs in $y_{1,2}$].

After obtaining the constant γ from Eq. (31), the quantities R and U are given as implicit functions of s by Eqs. (28) and (29).

The dimensional radius, longitudinal velocity, and wall thickness can be expressed, according to Eqs. (14) and (19), in terms of the functions R(s) and U(s):

$$\begin{gathered}\bar{r}(z) = \bar{r}_0 R[s(z/l)]; \; u(z) = u_0 U[s(z/l)]; \\ h(z) = h_0/\{R[s(z/l)]U[s(z/l)]\}.\end{gathered} \tag{32}$$

The final values of the wall thickness $h_\infty = h(+\infty)$ and the radius $\bar{r}_\infty = \bar{r}(+\infty)$ which describe the geometry of the drawn microcapillary can be expressed, using Eq. (27), in terms of the function R(s) for s = 1:

$$h_\infty = h_0\sqrt{(u_0/u_\infty)}K(P, \; Q, \; w); \; \bar{r}_\infty = \bar{r}_0\sqrt{(u_0/u_\infty)}[1/K(P, \; Q, \; w)], \tag{33}$$

where we have introduced the notation

$$K(P, \; Q, \; w) = \exp(-w/2)/R(1). \tag{34}$$

Substituting the boundary condition (27) into Eqs. (28) and (29), and using Eq. (31), we obtain

$$K(P, \; Q, \; w) = (1 - y_2)\exp[w(\gamma - 1)(3y_1 + y_2)/2(y_1 + y_2)] + y_2\exp(-w/2). \tag{35}$$

Hence, the problem of determining the geometry of the microcapillary (the final values of the wall thickness h_∞ and the radius $\bar{r}_\infty$) reduces to the numerical solution for each set of values of the parameters P, Q, and w of the transcendental equations (31) and (35), after which the quantities h_∞ and $\bar{r}_\infty$ are found from Eq. (33).

The functions R(s) and U(s) which, from Eq. (32) determine the profile of the jet, i.e., the quantities h(z) and $\bar{r}(z)$, can be found either by numerical solution of Eq. (28), or which is more convenient, by direct numerical integration of the system of equations (24) and (25) with the boundary conditions (26) and the known constant γ [obtained from Eq. (31)].

For $P \ll 1, Q \ll 1$ for the function K(P, Q, w) the following approximate expression [obtained by the method of successive approximations from Eqs. (24) and (25) taking Eqs. (26), (27), and (34) into account] holds:

$$K(P, \; Q, \; w) = 1 + Q - (P/2) + Q^2[(1/6) - (2/3w)] + PQ[(1/18) + (1/3w)] - (P^2/8) + 0(Q^2 + P^2) \tag{36}$$

[we used expressions (21) and (22) as the zeroth approximation for R(s) and U(s)].

We will determine under what conditions assumptions (8) hold in order that we may assume the pressure and the longitudinal velocity are constant over the transverse cross section. We will confine ourselves to the case of small values of P and Q so that we can use the zeroth approximation (21) as the solution for the longitudinal velocity. Substituting Eq. (21) and the expressions obtained for p using Eqs. (4) and (9) into the

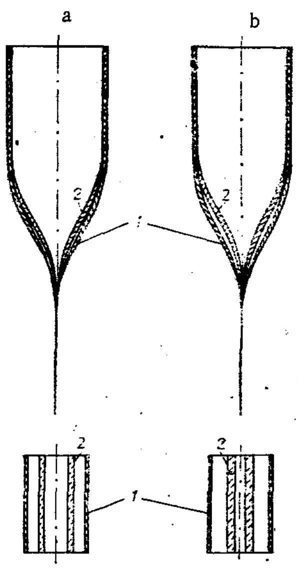

Fig. 2

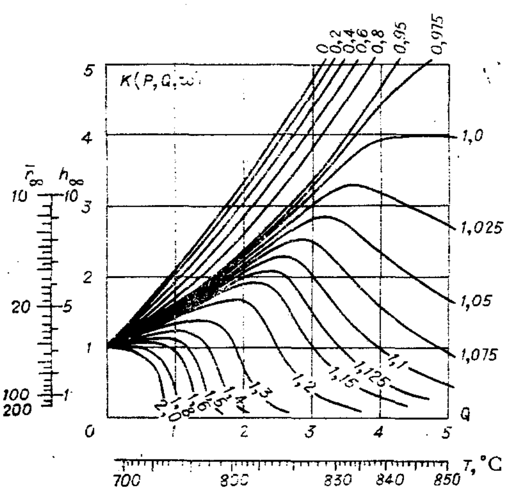

Fig. 3

Navier–Stokes equation (1) and assuming that $\partial/\partial z \sim 1/l$, since the unique characteristic length along the z axis is l, we find that assumptions (8) hold if $\bar{r}_0 h_0/l^2 \ll 1$.

In view of the fact that the fiber has thin walls this limitation is less strict than the previously made assumption that the slope of the boundary is small.

It is important that the final values of the wall thickness h_∞ and the radius $\bar{r}_\infty$ should be independent of the viscosity–temperature dependence and of the details of the temperature profile itself, and should depend only on the integral characteristic l/η_0 of the function $\eta(z)$, since according to Eq. (33) the quantities h_∞ and $\bar{r}_\infty$ depend on the viscosity only through the parameters P and Q which, of all the characteristics of the function $\eta(z)$, contain only the quantity l/η_0 [defined by Eq. (15)], which is the characteristic of the heating zone (an increase in the effective length of the heating zone l or an increase in the temperature leading to a reduction in the minimum viscosity η_0 corresponds to an increase in the characteristic l/η_0 of the heating zone).

Hence, for different temperature distributions, drawing of the microcapillary occurs for different jet profiles along the whole length of the blank until the microcapillary solidifies, but, if the integral characteristic of the viscosity distribution $\eta(z)$ is the same, the dimensions of the drawn microcapillary, i.e., the values of h_∞ and $\bar{r}_\infty$, will also be the same.

We will illustrate the dependence of the jet profile on the parameters P and Q for a fixed temperature distribution

$$T(z) = T_g + (T_m - T_g)/(1 + c_1 z^2),$$

where T_m is the maximum temperature, T_g is the vitrification temperature, and c_1 is a constant chosen to ensure the same value of l/η_0 in both cases with the following viscosity–temperature dependence:

$$\eta(T) = a \exp[b/(T - T_g)], \quad T > T_g$$

(a and b are constants which depend on the properties of the material).

The deformation of the jet profile with respect to the profile unperturbed by surface forces depends very much on the values of the parameters P and Q and the ratio between them [Fig. 2, where curve 1 is for the unperturbed profile with P = Q = 0; curve 2 is for the perturbed profile with P = 0, Q = 1 and P = 0, Q = 2 (Fig. 2a and b, respectively)]. The quantity Q is greater the greater the surface tension σ and the time the specimen resides in the heating zone l/u_0, and the lower the value of the minimum viscosity η_0 and the initial wall thickness h_0, which determines the thickness of the layer of the viscous liquid which resists the action of the surface tension forces.

The dimensions of the microcapillary, i.e., the quantities h_∞ and $\bar{r}_\infty$, are determined according to Eq. (33) by the dimensions of the blank h_0 and $\bar{r}_0$, the ratio of the drawing speed u_∞ to the feed rate u_0, and by the value of the function K(P, Q, w). When there are no surface forces ($\Delta p = \sigma = 0$ and, consequently, p = Q = 0) it follows from Eq. (36) that K = 1 and the final values of the wall thickness and the radius are proportional to the initial values of these quantities with a coefficient $\sqrt{u_0/u_\infty}$, which arises from the law of conservation of mass and which describes the reduction in the wall thickness and the radius while preserving the geometrical similarity of the cross sections of the microcapillary and the blank. The function $K(P, Q, w)$ describes the effect of the surface tension and pressure drop, which lead to a deviation from similarity depending on the ratio between them, or $K > 1$ in which case $h_\infty/\bar{r}_\infty > h_0/\bar{r}_0$, or $K < 1$ in which case $h_\infty/\bar{r}_\infty < h_0/\bar{r}_0$.

Figure 3 shows a family of curves which give K as a function of Q for $w = \ln(10^5)$ and different fixed values of the ratio P/Q. The curves corresponding to $w = \ln(10^4)$ and $w = \ln(10^6)$ practically coincide with the curves for $w = \ln(10^5)$, so that over this range of values of w we can neglect the explicit dependence of the function K on w (as previously, K depends implicitly on w, since w occurs in P and Q). The weak explicit dependence of K on w also follows from Eq. (36), since w only occurs in the terms which are quadratic in P and Q.

The family of curves in Fig. 3 together with Eq. (33) answers the question of how h_∞ and $\bar{r}_\infty$ depend on the dimensionless parameters P and Q.

To explain the dependence of the dimensions of the microcapillary on each of the dimensional parameters grouped in the two dimensionless quantities P and Q, it is necessary to analyze the results further. Figure 3 shows the dependence of h_∞ and $\bar{r}_\infty$ on the characteristic l/η_0 of the heating zone (in fact on its temperature) for fixed values of the pressure drop Δp, since, as is seen from Eq. (16), each of the quantities l/η_0 and Δp occurs only in one of the parameters Q and P/Q, respectively. The additional scales in Fig. 3 in which the maximum temperature of the heating zone T (°C) and the dimensions of the microcapillary h_∞ and $\bar{r}_\infty$ (μ) are plotted, are constructed for the following values of the remaining parameters: $h_0 = 0.1$ cm, $\bar{r}_0 = 1$ cm, $u_0 = 0.01$ cm/sec, $u_\infty = 10$ m/sec, $\sigma = 250$ dyn/cm, and $l = 5$ cm.

To obtain the direct dependence of the dimensions of the microcapillary on the other dimensional parameters of the process it is necessary when constructing the graphs to use as the argument and the parameter of the family not Q and P/Q, but other quantities (for example, the quantities P and Q, respectively, when investigating the dependence on Δp for fixed values of σ). Such graphs can be obtained from Fig. 3.

Within the framework of this problem we can only consider variations of the parameters that are slow compared with the characteristic time t_0 of the system. The upper estimate for t_0 is the ratio of the effective length of the heating zone l to the minimum longitudinal speed – the feed rate u_0: $t_0 = l/u_0$ (under practical conditions $t \sim 10^2$-10^3 sec). The problem of the sensitivity of the dimensions of the microcapillary to such slow variations of the parameters is considered using the above graph. It is seen from Fig. 3 that values of the parameters P and Q exist when $(\partial K/\partial Q)_{P/Q} = 0$. When the partial derivative is equal to zero this means that at this point the dimensions of the microcapillary are only slightly sensitive to a slow change in the characteristic l/η_0 of the heating zone for constant Δp. Note that, first, the extremal points correspond to $K > 1$, i.e., to greater or less collapse of the microcapillary, and, secondly, for large Q the curves in Fig. 3 are well separated from one another which corresponds to an extremely high sensitivity to changes in Δp. This obviously neutralizes any merit of these points such as the low sensitivity to slow variations of the quantity l/η_0 (i.e., the temperature of the heating zone).

Hence, modes of drawing which are acceptable from the point of view of sensitivity to slow variations of the parameters are those which correspond to $Q < 3$, while the parameter P is chosen so as to ensure that the process is extremal with respect to l/η_0.

Consequently, Eqs. (33) and the data in Fig. 3 give the dependence of the dimensions of the microcapillary on the dimensions of the blank and the parameters of the process. They can be used to choose the conditions under which the process is least sensitive to slow variations of the parameters. The effect of perturbations that are fast compared with the characteristic time t_0 must be considered separately.

The authors thank B. Z. Katsenelenbaum and A. D. Shatrov for useful discussions, A. I. Leonov for a number of observations, and I. V. Aleksandrov, T. V. Bukhtiarov, and A. A. Dyachenko for discussing the results at various stages.

LITERATURE CITED

1. L. R. Glicksman, Trans. ASME, J. Basic Eng., Sept. (1968).
2. M. S. Aslanova and S. S. Gordon, Mekh. Polim., No. 4, 599 (1973).